Journal of Engineering Mathematics 18 (1984) 235–246.

Unsteady flow from a source in a rotating fluid

S.H. SMITH

Department of Mathematics, University of Toronto, Toronto M5S 1A1, Canada

(Received March 23, 1984)

Summary

A source, which is situated on the axis of a rotating fluid, commences to expel fluid with constant rate at the time $t = 0$. We describe how the geostrophic forces lead to the formation of a narrow column along the axis, before the eventual development of the viscous Stewartson column along the axis, and how the final steady state is achieved. An understanding of the role of the non-linear inertial forces in the neighbourhood of the source is given. The results are also extended by considering the effect of placing the source between two infinite discs situated perpendicular to the axis of rotation.

1. Introduction

The first important results concerning the transient development of flows in a rotating fluid were included in the now classical paper by Greenspan and Howard [1]. In this work they discovered the basic process whereby fluid in a cylindrical container is spun-up, after an impulsive rotation is imparted to the cylinder, in a time which is $O(E^{-1/2})$ where E is the Ekman number; if diffusion alone acted the time would have been the much longer $O(E^{-1})$. For simplicity, the majority of their discussion centred on the geometry with two infinite discs, but in the process they were able to show the crucial role of the Ekman layers in the whole process. Their's was a linear theory, but it was easily seen that the basic conclusion is unaltered when the Rossby number is finite.

The next major step, to understand the effect of the side wall of the cylinder during spin-up, was presented by Wedemeyer [2]. He developed a highly approximate non-linear analysis to show the possibility of a cylindrical wave front, which propagates from the side wall into the fluid, across which there is a discontinuity in shear; experimental evidence justifies this model. Improvements to Wedemeyer's results have been presented, and many accurate numerical calculations have been performed (cf. Warn-Varnas, Fowlis, Piacsek and Lee [3]), all of which help to confirm the basic idea.

One area where there is less certainty concerns how the different boundary layers on the side wall are formed; in fact, it is still not clearly known what the thicknesses of these layers are when the Rossby number is $O(1)$. A first step in this understanding was the paper by Smith [4], where an attempt was made to extent the linear theory to the parameter domain where inertial forces need be included. However, even for the linear regime, the analytical results were complicated, being left sometimes as double integrals, and not particularly straightforward to interpret. The purpose of the present paper is to give some of these procedures a more precise description..

The model we investigate is that of a point source of fluid, situated on the axis of a rotating fluid, which commences to act impulsively at the time $t = 0$. Therefore, this is a fundamental solution for unsteady flows in a rotating fluid, and because there is no natural length scale, the solution found is valid for all source strengths ϵ. It is shown that the non-linear forces act initially within a distance $O(\epsilon^{1/3})$ from the source, but that this domain grows continuously in the direction parallel to the axis up to a distance which is $O(\epsilon E^{-1})$ by the time $O(\epsilon^{2/3}E^{-1})$ has elapsed for the final steady state to be developed. (A slightly different description follows when the source strength is very weak when $\epsilon \ll E^{3/2}$, but this is of lesser interest.) Outside this domain the flow is governed by linear equations throughout the development, and these can be completely solved. The main interest is in the formation of the column, through the action of inviscid geostrophic forces, when the time after the impulsive beginning is small compared to $O(E^{-1/3})$. The flow can be described by a single similarity variable rt/z (r and z are the non-dimensional radial and axial distances, respectively), and the governing equations are linear, fourth-order differential equations which can be formally solved in terms of Bessel and Struve functions.

When the source is placed between parallel discs, which otherwise rotate with the fluid as a solid-body, then it is necessary to restrict the source strength by $\epsilon \ll E$ to ensure a linear domain. We find (Section 5), that the time for the potential vortex to form in the fluid is the spin-up time $O(E^{-1/2})$, as it is for the column with radius $O(E^{1/4})$ to be formed. During the time that the inviscid column is being developed the flow can be described by an infinite set of sources which are reflections of the basic source in the discs; however, because of the columnar nature of the flow, the dominant contribution is given by just the first reflection.

This fundamental solution is of interest in itself, but we also find (Section 6) that the ideas presented are easily extended to the situation when he point source is replaced by a ring source at $r = 1$, $z = 0$ (the axial symmetry is retained). During the early stages of spin-up in the circular cylinder, the Ekman layers transport mass along the discs to the corner regions, which then propagates into the interior as if flowing from a ring source in the corner (cf. Moore and Saffman [5]); this was also clearly evident from the time dependent description of Smith [4], and that this analogy was relevant for all $\epsilon \ll 1$. This observation motivates the extension. An inviscid layer forms along $r = 1$, with properties very similar to those in the column described earlier, and with the same balance of forces. The changes in the analysis are mostly simplifications, and are mathematical in nature rather than physical. It is apparent that all vertical shear layers in a rotating fluid pass through this stage during the beginning of a spin-up process.

2. The fundamental solution

A fluid rotates with constant angular velocity Ω, and we consider the flow due to a source placed at the point O on its axis of rotation; the source commences to expel fluid at a constant rate from the time $t = 0$. Let a be the reference length, and ar, az represent radial and axial distances from the origin O; the physical time is measured by $\Omega^{-1}t$. We write $\Omega au(r, z, t)$, $\Omega av(r, z, t)$ and $\Omega aw(r, z, t)$ for the radial, azimuthal and axial velocities respectively; the pressure is $\rho\Omega^2a^2p(r, z, t)$ when the constant density of the fluid is ρ. If the strength of the source ϵ is such that the equations can be linearised, then writing

$u = \epsilon U$, $v = r + \epsilon V$, $w = \epsilon W$, $p = \frac{1}{2}r^2 + \epsilon P$, we have

$$U_r + \frac{1}{r}U + W_z = \frac{1}{r}\delta(r)\delta(z)H(t), \tag{2.1}$$

$$U_t - 2V = -P_r + E\left(U_{rr} + \frac{1}{r}U_r - \frac{1}{r^2}U + U_{zz}\right), \tag{2.2}$$

$$V_t + 2U = E\left(V_{rr} + \frac{1}{r}V_r - \frac{1}{r^2}V + V_{zz}\right), \tag{2.3}$$

$$W_t = -P_z + E\left(W_{rr} + \frac{1}{r}W_r + W_{zz}\right); \tag{2.4}$$

the parameter E is the Ekman number, defined as $\nu/\Omega a^2$, where ν is the constant kinematic viscosity of the fluid. In all which follows, $E \ll 1$. The only conditions to impose are that the velocities U, V, $W \to 0$ as r, $z \to \infty$. This is seen to be the transient problem corresponding to the steady state situation considered by Smith [6].

The solution to the equations (2.1)–(2.4) follow on taking a Laplace transform in t, a Fourier cosine or Fourier sine transform in z, and a Hankel transform of order zero or unity in r; for example, with $V(r, z, t)$ and $W(r, z, t)$ we set

$$\overline{V}(k, \alpha, s) = k\int_0^\infty e^{-st}dt\int_0^\infty \cos\alpha z dz\int_0^\infty rV(r, z, t)J_1(kr)dr, \tag{2.5a}$$

$$\overline{W}(k, \alpha, s) = k\int_0^\infty e^{-st}dt\int_0^\infty \sin\alpha z dz\int_0^\infty rW(r, z, t)J_0(kr)dr. \tag{2.5b}$$

The calculations are straightforward, and show

$$\begin{aligned} &k\overline{U} + \alpha\overline{W} = ks^{-1}, & &s\overline{U} - 2\overline{V} = k\overline{P} - E(k^2+\alpha^2)\overline{U}, \\ &s\overline{V} + 2\overline{U} = -E(k^2+\alpha^2)\overline{V}, & &s\overline{W} = \alpha\overline{P} - E(k^2+\alpha^2)\overline{W}, \end{aligned} \tag{2.6}$$

from which we can solve for $\overline{V}$ to find

$$\overline{V} = -\frac{2k^2\{E(k^2+\alpha^2)+s\}}{s\left[\{E(k^2+\alpha^2)+s\}^2(k^2+\alpha^2)+4\alpha^2\right]}. \tag{2.7}$$

Similar expressions follow for $\overline{U}$, $\overline{W}$ and $\overline{P}$. We note that these are exact solutions of the linear equations (2.1)–(2.4), and are valid for all values of E.

The inverse transform for (2.5) becomes

$$V(r, z, t) = \frac{1}{\pi^2 i}\int_{c-i\infty}^{c+i\infty} e^{st}ds\int_0^\infty \cos\alpha z d\alpha\int^\infty J_1(kr)\overline{V}(k, \alpha, s)dk. \tag{2.8}$$

There are two major approximations which can be calculated from the triple integral (2.8) corresponding to the two main time scales during which different forces are in balance. In the early stages of the flow development it can be expected that inviscid terms alone dominate, and that the viscous terms come into prominence only as the flow from the source becomes more concentrated and the shear increases. Hence, in the first instance, we neglect the viscous terms in $\bar{V}$ by letting $E \to 0$; two of the integrals in (2.8) can be evaluated to leave

$$V \simeq -\frac{1}{\pi i r^2}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s}\frac{(s^2+4)\mathrm{d}s}{\left\{s^2(1+z^2/r^2)+4\right\}^{3/2}}. \tag{2.9}$$

This integral (2.9) can be evaluated as the series

$$V \simeq -\frac{2rt}{R^2}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!^2}\left(\frac{r}{R}\right)^{2n-2}\left\{2-\left(2+\frac{1}{n}\right)\frac{r^2}{R^2}\right\}t^{2n}\right], \tag{2.10}$$

where $R^2 = r^2 + z^2$; this series is convergent for all values of r, z, t. General though this result is, it does not reveal the dominant feature, which is the formation of the narrow columnar region along the axis of rotation which represents the response of the flow to the Taylor-Proudman theorem. Nevertheless, the role of the variable rt/R is suggested, and when we make the further approximation $r/z \ll 1$ in the integral (2.9) it follows that

$$V \simeq -\frac{1}{r^2}\int_0^{\sigma}\kappa J_1(\kappa)\mathrm{d}\kappa = -\frac{\pi\sigma}{2r^2}\{J_1(\sigma)H_0(\sigma)-J_0(\sigma)H_1(\sigma)\}, \tag{2.11}$$

(cf. Erdelyi et al. [7], p. 236), where $J_n(\sigma)$ are Bessel functions and $H_n(\sigma)$ are Struve functions (c.f. Abramowitz and Stegun [8]); the similarity variable σ is given by

$$\sigma = \frac{2rt}{z}. \tag{2.12}$$

The expression (2.11) must be seen as the approximation to the inverse transform (2.7), (2.8) as $E \to 0$ for $\sigma = O(1)$ when $r/z \ll 1$. The thin column along the axis of rotation forms through the constraining effect of the linear geostrophic forces, and can be considered as developed once $t \gg 1$, though the diameter of the column continues to narrow as $O(t^{-1})$ a finite distance from the source.

The expansion (2.9), and consequently (2.10) is a solution of the set of equations (2.1)–(2.4) with the viscous terms absent; also, making the approximation r/z small is equivalent to further neglecting the radial velocity term U_t in (2.2). Hence, (2.11) is a solution of the equation of continuity (2.1) plus the momentum equations

$$2V = P_r, \quad V_t + 2U = 0, \quad W_t = -P_z;$$

eliminating U, W and P (and ignoring the delta function in (2.1)) shows

$$V_{rrtt} + \frac{1}{r}V_{rtt} - \frac{1}{r^2}V_{tt} + 4V_{zz} = 0. \tag{2.13}$$

We consider this, and the corresponding equation for the stream function in the next section.

The second simplification to (2.7), (2.8) follows from retaining the viscous terms, but taking $\partial/\partial z \ll \partial/\partial r$ in the columnar region. This possibility follows from the absence of any Ekman layer in the flow, and enables us to set $\alpha^2 \ll k^2$ in the inverse transform (2.7). The result shows, after completing two of the integrations, that

$$V \simeq -\frac{1}{r^2}\int_0^{\sigma} \kappa J_1(\kappa) e^{-(1/2)\kappa^3\mu} d\kappa, \quad z > 0, \tag{2.14}$$

where

$$\mu = \frac{Ez}{r^3}; \tag{2.15}$$

and $\mu = O(1)$; a similar result follows for $z < 0$ because $V(r, z, t)$ is symmetric about $z = 0$. In the particular case $\mu \to 0$, then (2.11) is recovered.

Once we observe that for μ, $\sigma = O(1)$, then $\lambda = E^{1/3}tz^{-2/3}$ is finite, it becomes possible to delineate the different processes which lead to the formation of what we call the Stewartson column. For $t = O(1)$ there is the initial adjustment to the introduction of the source flow, which leads, through the action of the Coriolis force, to the inertial column which is defined for large t by the similarity variable $\sigma = O(1)$; the radius of the column is $O(t^{-1})$ when z is finite. As the time increases to permit $\lambda = O(1)$, which indicates $t = O(E^{-1/3})$ for finite z, viscous forces begin to act and the column is defined by $\mu = O(1)$, or $r = O(E^{1/3})$ when $z = O(1)$. The final steady state, as previously described in [6], is attained as $\sigma \to \infty$, or $E^{1/3}t \to \infty$ when z is finite.

So far, only the analytical approximations for the azimuthal velocity V have been given; the corresponding expressions for the other velocity components are little different with, for example

$$W \simeq \frac{1}{r^2}\sigma J_1(\sigma), \quad \text{for } \sigma = O(1), \tag{2.16}$$

corresponding to (2.11), and

$$W \simeq \frac{1}{r^2}\int_0^{\sigma} \kappa J_0(\kappa) e^{-(1/2)\kappa^3\mu} d\mu, \quad z > 0, \tag{2.17}$$

for $\mu = O(1)$, corresponding to (2.14). From both these integrals it can be confirmed that the total flux out of the source is constant through calculating $\int_0^\infty rW dr$.

We now continue by considering several different extensions based on these results.

3. Similarity solutions for the columnar region

The equation for the azimuthal velocity $V(r, z, t)$ during the initial stages of the development of the column when $\sigma = O(1)$ is given by (2.13); the same equation is also satisfied by the radial velocity $U(r, z, t)$. When we introduce a stream function $\Psi(r, z, t)$ by $U = -r^{-1}\Psi_z$ and $W = r^{-1}\Psi_r$, then Ψ satisfies

$$\Psi_{rrtt} - \frac{1}{r}\Psi_{rtt} + 4\Psi_{zz} = 0. \tag{3.1}$$

Finally, both W and P satisfy the same equation

$$W_{rrtt} + \frac{1}{r} W_{rtt} + 4W_{zz} = 0. \tag{3.2}$$

We now consider the general question of the existence of solutions to the equation (3.1) of the form

$$\Psi = r^a f(\sigma), \quad \sigma = \frac{2rt}{z},$$

where a is a constant. Substitution into (3.1) leads to the ordinary differential equation

$$\sigma^2 f^{\text{iv}} + (2a+3)\sigma f''' + a(a+2)f'' + \sigma^2 f'' + 2\sigma f' = 0. \tag{3.3}$$

It is now possible to write $f'(\sigma) = \sigma^{-a} k(\sigma)$, and integrate the resultant equation once, to find

$$\sigma^2 k'' + \sigma k' + (\sigma^2 - 1)k = A\sigma^a, \tag{3.4}$$

where A is the constant of integration. The equation (3.4) has three linearly independent solutions, two of which are the Bessel functions $J_1(\sigma)$ and $Y_1(\sigma)$ for all values of a; the third solution can generally be written as the Lommel function (cf. Erdelyi et al. [9], p. 40) $S_{a-1,1}(\sigma)$, which can also be expressed in terms of the particular hypergeometric function $\sigma^a_1F_2(1; \frac{1}{2}(1+a); \frac{1}{2}(3+a); -\frac{1}{4}\sigma^2)$. In the special case when $a = 0$, which is the appropriate value for the stream function in this problem, this third solution is just $H_1(\sigma) - 2\pi^{-1}$, where $H_1(\sigma)$ is the Struve function. Hence, the four similarity solutions for $\Psi(r, z, t)$ are proportional to $J_0(\sigma)$, $Y_0(\sigma)$, $H_0(\sigma)$ and 1.

The same general discussion for the equations (2.13) and (3.2) is also possible, though we omit the details here, except to report that for (2.13) the four similarity solutions for $r^2V(r, z, t)$ are proportional to $\int_0^\sigma \kappa J_1(\kappa)\mathrm{d}\kappa$, $\int_0^\sigma \kappa Y_1(\kappa)\mathrm{d}\kappa$, $\int_0^\sigma \kappa H_1(\kappa)\mathrm{d}\kappa$ and 1.

The conditions to be satisfied for Ψ are that $\Psi = 0$ for $\sigma = 0$ and $\Psi \to \frac{1}{2}$ as $\sigma \to \infty$ - which ensures that mass is conserved. Consequently

$$\Psi = \tfrac{1}{2}\{1 - J_0(\sigma)\} \tag{3.5}$$

is the correct solution, from which (2.16) follows. Similarly, for V, the result (2.11) is confirmed.

4. The non-linear contribution

As $r, z \to 0$ the velocities in the azimuthal plane become large, and so it is inevitable that there is a region close to the origin where non-linear terms must be included. The full Navier-Stokes equations for $r, z \neq 0$ are

$$u_r + r^{-1}u + w_z = 0, \tag{4.1}$$

$$u_t + uu_r + wu_z - r^{-1}v^2 = -p_r + E\left(u_{rr} + r^{-1}u_r - r^{-2}u + u_{zz}\right), \tag{4.2}$$

$$v_t + uv_r + wv_z + r^{-1}uv = E\left(v_{rr} + r^{-1}v_r - r^{-2}v + v_{zz}\right), \tag{4.3}$$

$$w_t + uw_r + ww_z = -p_z + E\left(w_{rr} + r^{-1}w_r + w_{zz}\right). \tag{4.4}$$

Now the linear analysis for small ϵ has shown that we can introduce functions Ψ_0 and V_0 and be able to write the stream function $\psi = \Psi_0(\sigma)$ and the azimuthal velocity $v = r + \epsilon r^{-2}V_0(\sigma)$ in the column where r is small with $\sigma = O(1)$. From the expression for v it is clear immediately that the linearization requires $r \gg \epsilon^{1/3}$. Further, when these expressions for ψ and v are substituted into the angular momentum equation (4.3), the non-linear terms are seen to be $O(\epsilon^2 r^{-4} z^{-1})$, and the linear terms are $O(\epsilon r^{-1} z^{-1})$ when $\sigma = O(1)$. These have equal order when $r = O(\epsilon^{1/3})$; a similar conclusion follows from the other momentum equations. Consequently, the size of the domain where the non-linear forces must be considered as $r \to 0$, maintaining $\sigma = O(1)$, requires $r = O(\epsilon^{1/3})$; the axial dimension of this domain is $z = O(\epsilon^{1/3}t)$ as t becomes large. The viscous terms are absent and the radial momentum equation reduces to the centrifugal force balance.

Secondly, for the viscous Stewartson column where both μ, $\sigma = O(1)$, it follows that $u = O(\epsilon E^{-1/3} z^{-4/3})$ and $v - r$, $w = O(\epsilon E^{-2/3} z^{-2/3})$ from (2.14), (2.16). Consequently, the inertial terms in the angular momentum equation are $O(\epsilon^2 E^{-4/3} z^{-7/3})$, and the linear terms are $O(\epsilon E^{-1/3} z^{-4/3})$, so that the domain where non-linear terms must be included is defined by $r = O(\epsilon^{1/3})$, $z = O(\epsilon E^{-1})$; this is unchanged from the domain found for the steady-state case (cf. Smith [6]) by equivalent arguments. The scalings and the equations are also the same as given there, with the addition that the transient terms are included when the order of magnitude for the time is $O(\epsilon^{2/3}E^{-1})$. The terms neglected are small compared to those retained when $\epsilon \gg E^{3/2}$ – a condition we retain throughout this section.

It is now clear that for finite times the non-linear effects are concentrated in the region where r, $z = O(\epsilon^{1/3})$, but as t increases this region grows in the vertical direction as $z = O(\epsilon^{1/3}t)$, occupying the full non-linear domain with $z = O(\epsilon E^{-1})$ when $t = O(\epsilon^{2/3}E^{-1})$. However, this time is still small compared with $O(E^{-1/3}z^{2/3})$, the time required for the steady state to be reached in the linear domain where $\mu = O(1)$, $z \gg \epsilon E^{-1}$. Hence, the flow is quasi-steady in the non-linear domain once $t \gg \epsilon^{2/3}E^{-1}$, while the steady in the non-linear domain once $t \gg \epsilon^{2/3}E^{-1}$, while the steady state is being attained elsewhere.

The remaining question concerns the formation of the shear layer in the region where r, $z = O(\epsilon^{1/3})$. When the flow is steady, Barua [10] and Squire [11] showed how the outflow from the source is restricted within a cylindrical column along the axis of rotation; this is separated from the remainder of the flow by a shear layer with thickness $O(E^{1/2})$. They attempted to calculate its radius by different procedures, but because it was treated as a local flow the problem as posed was indeterminate, and an extra assumption was necessary to give the precise value. Their conclusions agreed qualitatively with each other, although the precise value for the radius did differ. For the unsteady flow, we define $r = \epsilon^{1/3}\rho$, $z = \epsilon^{1/3}\zeta$, $u = \epsilon^{1/3}\mathcal{U}$, $v = \epsilon^{1/3}\mathcal{V}$, $w = \epsilon^{1/3}\mathcal{W}$, $p = \epsilon^{2/3}\mathcal{P}$, for the resulting inviscid equations

$$\begin{aligned} &\mathcal{U}_\rho + \rho^{-1}\mathcal{U} + \mathcal{W}_\zeta = 0, &&\mathcal{U}_t + \mathcal{U}\mathcal{U}_\rho + \mathcal{W}\mathcal{U}_\zeta - \rho^{-1}\mathcal{V}^2 = -\mathcal{P}_\rho, \\ &\mathcal{V}_t + \mathcal{U}\mathcal{V}_\rho + \mathcal{W}\mathcal{V}_\zeta + \rho^{-1}\mathcal{U}\mathcal{V} = 0, &&\mathcal{W}_t + \mathcal{U}\mathcal{W}_\rho + \mathcal{W}\mathcal{W}_\zeta = -\mathcal{P}_\zeta; \end{aligned} \tag{4.5}$$

the error in neglecting the viscous terms is $O(E\epsilon^{-2/3})$.

The equations (4.5) can, in principle, be solved as power series in t – the non-linear equivalent of the representation (2.10) – but this would give no hint of the development of the shear layer. It can, however, be seen that there does exist a solution for this set of equations, which utilizes their full non-linearity, in terms of some similarity variable, to indicate the formation of the shear layer. Although this discussion would perhaps be clearer following that given in Section 6, we complete it here in its more natural context. Specifically, we consider a solution for large t to the equation (4.5) of the form

$$\begin{aligned} &\mathscr{U}=\{\alpha(t)\}^{-1}\mathscr{F}(\eta,\zeta), && \mathscr{V}=\mathscr{G}(\eta,\zeta),\\ &\mathscr{W}=\mathscr{H}(\eta,\zeta), && \mathscr{P}=\{\alpha(t)\}^{-1}\mathscr{R}(\eta,\zeta) \end{aligned} \tag{4.6}$$

where $\eta=\{\rho-\rho_0(\zeta)\}\alpha(t)$ is the similarity variable for some (as yet unknown) function $\alpha(t)$ which grows without bound as t increases; ρ and ζ are finite, and $\rho_0(\zeta)$ gives the radius of the shear layer when t is infinite, as considered by Barua and Squire. These representations ensure that (i) both the azimuthal and axial velocities are finite as t increases, (ii) the equation of continuity is satisfied, and (iii) the pressure gradient and centrifugal force balance in the radial momentum equation as necessary for consistency. When the expressions (4.6) are substituted into the third and fourth of the equations (4.5) it follows that in both equations the non-linear terms are $O(1)$ and the transient terms are $O(\dot{\alpha}/\alpha)$; the pressure gradient is $O(\alpha^{-1})$, all for large t. Consequently, there is a balance between the inertial and transient terms when $\dot{\alpha}/\alpha$ is finite, which requires $\alpha(t)\propto e^{ct}$ for some positive constant c; the pressure gradient is then negleigible in the axial momentum equation. That is, the development of the shear layer is measured by the variable

$$\eta=\{\rho-\rho_0(\zeta)\}e^{ct} \tag{4.7}$$

as t increases, and the dominant terms in (4.5) during this period are $\mathscr{U}_\rho+\mathscr{W}_\zeta=0$, $\rho^{-1}\mathscr{V}^2=\mathscr{P}_\rho$, $\mathscr{V}_t+\mathscr{U}\mathscr{V}_\rho+\mathscr{W}\mathscr{V}_\zeta=0$, $\mathscr{W}_t+\mathscr{U}\mathscr{W}_\rho+\mathscr{W}\mathscr{W}_\zeta=0$. Clearly more information regarding the geometry of the shear layer is required before any useful further progress is possible, but the existence of the similarity variable (4.7) lends weight to the possibility that a similar mechanism for the development of the shear layer is present in this situation as for the Stewartson layer in the case of the source described in Section 6. After a large enough time has elapsed, the layer is sufficiently narrow for viscous forces to be included, and the final steady state as discussed by Barua and Squire is attained.

One last point to observe is that the preceding comments on the non-linearity are valid for all values of $\epsilon \gg E^{3/2}$ – even for finite or large ϵ. The length scale for the problem is, in fact, defined partially by the strength of the source ϵ, and although many of the calculations only have point when ϵ is small, their validity goes beyond this restriction.

5. A source between parallel discs

An extension to the fundamental solution calculated in Section 2 is to consider the effects when the source is placed between two discs, which are normal to the axis, and rotate with the fluid at constant angular velocity Ω. For simplicity we assume the source is situated at the origin, midway between the discs; however, the main conclusions of this section are independent of its location (cf. Smith [6]). The strength of the source is small enough for the flow to be linear for all points a finite distance from the source, and this requires $E^{3/2} \ll \epsilon \ll E$, following the results of the previous section.

The azimuthal velocity $V(r, z, t)$ for the fundamental solution, which follows from neglecting the terms $E\partial^2/\partial z^2$ in the equations (2.2)–(2.4), and then evaluating the inverse Fourier transform, shows

$$V \simeq -\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{st}}{s}\mathrm{d}s\int_0^\infty \frac{kJ_1(kr)}{\{(Ek^2+s)^2+4\}^{1/2}}\mathrm{e}^{-\theta z}\mathrm{d}k \tag{5.1}$$

where

$$\theta = k(Ek^2+s)\{(Ek^2+s)^2+4\}^{-1/2};$$

equivalent expressions follow for U, W, P. consequently, when the discs are positioned along $z = \pm d$, then we can write

$$V = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{st}}{s}\mathrm{d}s$$

$$\times\int_0^\infty\left[-2k\mathrm{e}^{-\theta z}+2C(k,s)\cosh\theta z\right]\{(Ek^2+s)^2+4\}^{-1/2}J_1(kr)\mathrm{d}k, \tag{5.2}$$

$$W = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{st}}{s}\mathrm{d}s$$

$$\times\int_0^\infty\left[k\mathrm{e}^{-\theta z}+C(k,s)\sinh\theta z\right]J_0(kr)\mathrm{d}k, \tag{5.3}$$

$$P = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{st}}{s}\mathrm{d}s$$

$$\times\int_0^\infty\left[\mathrm{e}^{-\theta z}-k^{-1}C(k,s)\cosh\theta z\right]\{(Ek^2+s)^2+4\}^{1/2}J_0(kr)\mathrm{d}k, \tag{5.4}$$

for $z \geqslant 0$, with $C(k, s)$ representing some unknown function, as the general solution for the flow between the discs except for the thin Ekman layer of thickness $O(E^{1/2})$ along the discs $z = \pm d$. The expression for $C(k, s)$ is found through satisfying the Ekman condition

$$W = -\tfrac{1}{4}E^{1/2}\left(P_{rr}+r^{-1}P_r\right) \quad \text{on} \quad z = d.$$

It follows that

$$C(k,s) = -\frac{\left[k-\frac{1}{4}E^{1/2}k^2\{(Ek^2+s)^2+4\}^{1/2}\right]e^{-\theta d}}{\sinh\theta d+\frac{1}{4}E^{1/2}k\{(Ek^2+s)^2+4\}^{1/2}\cosh\theta d} \tag{5.5}$$

exactly, to complete the solution.

Approximations can now be taken to describe the flow for the different time scales during which the development passes. To begin, we take $r = O(1)$ and $t = O(E^{-1/2})$ to find from (5.2) and (5.5) that

$$V \simeq -2E^{-1/2}r^{-1}\left(1 - e^{-E^{1/2}t/d}\right); \tag{5.6}$$

in the interior the potential vortex requires the time $O(E^{-1/2})$ to develop in a manner very similar to that present in spin-up. The fluid from the source is transported away in the Ekman layers, and it is the response of the fluid in the interior to this outflow which leads to the decreased azimuthal velocity.

For the time range $1 \ll t \ll E^{-1/3}$, the inviscid column is being formed, and when we set $rt = O(1)$ in the integral (5.2), using (5.5), it follows that

$$V \simeq -\frac{\pi t}{rz}\mathscr{I}\left(\frac{2rt}{z}\right) - \frac{\pi t}{r(2d-z)}\mathscr{I}\left(\frac{2rt}{2d-z}\right), \quad 0 \leqslant z < d, \tag{5.7}$$

where $\mathscr{I}(x)$ is defined in terms of Bessel and Struve functions as

$$\mathscr{I}(x) = J_1(x)H_0(x) - J_0(x)H_1(x);$$

a similar result follows for $-d \leqslant z \leqslant 0$. The first term comes from the fundamental solution (2.11), and the second term is its reflection in the plane $z = d$. Actually, there is an infinite set of reflected terms present, as could be expected from the essentially inviscid nature of the flow, but because of the columnar nature of the flow, with r small, this second term indicating just one reflection dominates all others. The corresponding expression for the axial velocity is

$$W \simeq \frac{2t}{rz}J_1\left(\frac{2rt}{z}\right) - \frac{2t}{r(2d-z)}J_1\left(\frac{2rt}{2d-z}\right), \quad 0 \leqslant z < d,$$

to show zero axial velocity on $z = d$.

When $t = O(E^{-1/3})$, the viscous $E^{1/3}$-column is formed along the axis; here the integrals can only be slightly simplified to show, for example,

$$V \simeq -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s}ds\int_0^\infty \frac{k\cosh\frac{1}{2}ks(z-d)}{\sinh\frac{1}{2}ksd}J_1(kr)dk, \quad 0 \leqslant z < d.$$

Finally, for the larger times $O(E^{-1/2})$, the $E^{1/4}$-column is formed, growing out of the narrower $E^{1/3}$-column, but again, the azimuthal velocity can be evaluated no further than

$$V \simeq -\frac{2}{r}E^{-1/2}\left(1 - e^{-E^{1/2}t/d}\right)$$
$$+\frac{E^{-3/4}}{\pi i d}\int_{c-i\infty}^{c+i\infty}\frac{e^{E^{1/2}tx}}{x(x+d^{-1})^{1/2}}K_1\left\{(x+d^{-1})^{1/2}E^{-1/4}r\right\}dx,$$

with $r = O(E^{1/4})$, $t = O(E^{-1/2})$.

6. Ring source

If the source of fluid takes the form of a circle with unit radius in the non-dimensional co-ordinate system, and the flow from this ring is constant throughout its length, then the mathematical formulation is the same as that given in Section 2, except that we just replace the delta function $\delta(r)$ by $\delta(r-1)$ in the equation of continuity (2.1). This new situation can be seen as giving a simplified model for the early development of the flow due to the collision of fluid from the Ekman layers at the commencement of spin-up (cf. Smith [4]). The only result of this change of delta functions from the solution given previously is that now the transform functions $\overline{U}$, $\overline{V}$, $\overline{W}$, $\overline{P}$, as given through (2.5), (2.6) must now be multiplied by the Bessel function $J_0(k)$.

All the calculations of Sections 2–5 could, of course, be repeated; however, here we focus only on those of greatest interest. When $(r-1)\,(Ez)^{-1/3}$ and $(r-1)t$ are finite, then the approximations lead to

$$V \simeq -\frac{1}{\pi}\int_0^{2t/z} \sin k(r-1) \mathrm{e}^{-(1/2)Ek^3 z} \mathrm{d}k, \quad z > 0, \tag{6.1}$$

which corresponds to (2.14) for the Stewartson column. The equivalent expression for the axial velocity shows

$$W \simeq -\frac{1}{\pi}\int_0^{2t/z} \cos k(r-1) \mathrm{e}^{-(1/2)Ek^3 z} \mathrm{d}k, \quad z > 0, \tag{6.2}$$

it is clear that these represent the transient development of the similarity solutions given by Moore and Saffman [5].

As this layer is being developed, during times where $1 \ll t \ll E^{-1/3}$, geostrophic forces lead to the existence of the distinct limit as $(r-1)t = O(1)$. In fact, directly from (6.1), (6.2) it can be seen that

$$V \simeq -\frac{1}{\pi(r-1)}\left\{1-\cos\left(\frac{2(r-1)t}{z}\right)\right\},$$

$$W \simeq -\frac{1}{\pi(r-1)}\sin\left(\frac{2(r-1)t}{z}\right), \tag{6.3}$$

when we explicitly require $(r-1)t/z = O(1)$; these are attractively simple formulae for the inviscid state of the development of the $E^{1/3}$ Stewartson layer.

The equation satisfied by both the expressions (6.3) is $X_{rrtt} + 4X_{zz} = 0$, and when $X(r, t, z) = (r-1)^{-1} f(\xi)$, $\xi = 2(r-1)t/z$, then

$$\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + 2\right)(f''' + f') = 0.$$

The four linearly independent solutions for this equation are $f(\xi) \propto 1$, $\cos \xi$, $\sin \xi$ and $\int_\infty^\xi \mathrm{d}v \int_\infty^v u^{-1} \cos(u-v)\mathrm{d}u$.

The non-linear terms need to be included in the development of this layer where $\xi = O(1)$ when $r - 1 = O(\epsilon^{1/2})$, and so the domain grows as $z = O(\epsilon^{1/2} t)$. The time required for the $E^{1/3}$ Stewartson layer to form is $O(\epsilon E^{-1})$, and the inertial forces are present in the region where $r-1=O(\epsilon^{1/2})$, $z=O(\epsilon^{3/2}E^{-1})$.

When the ring source is placed midway between two horizontal discs along $z=\pm d$ the azimuthal velocity develops during the spin-up time $\tau=E^{1/2}t=O(1)$, with

$$V=\begin{cases}0, & r<1,\\ -2E^{-1/2}r^{-1}(1-e^{-\tau/d}), & r>1.\end{cases}$$

Within the $E^{1/4}$-layer the expressions can be simplified for the azimuthal velocity, to

$$V\simeq-\tfrac{1}{2}E^{-1/2}\left[4(1-e^{\tau/d})+2e^{-\tau/d}\operatorname{erfc}\left(\frac{\omega}{2\sqrt{\tau}}\right)\right.$$
$$\left.-e^{\omega/\sqrt{d}}\operatorname{erfc}\left(\frac{\omega}{2\sqrt{\tau}}+\frac{\tau}{d}\right)-e^{-\omega/\sqrt{d}}\operatorname{erfc}\left(\frac{\omega}{2\sqrt{\tau}}-\frac{\tau}{d}\right)\right],$$

where $\omega=(r-1)/E^{1/4}$, for $\omega>0$; a similar expression follows for $\omega<0$.

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